QUIZ #4 – Solutions
Each problem is worth 5 points

15 points total

1.

The auxiliary equation is $0 = m^4 + 5m^2 + 4 = (m^2+1)(m^2+4)$ with solutions $m = \pm i, \pm 2i$. A general solution of the differential equation is therefore $y(x) = C_1 \cos x + C_2 \sin x + C_3 \cos 2x + C_4 \sin 2x$.

2.

The auxiliary equation is $0 = 2m^2 + 16m + 82$ with solutions $m = -4 \pm 5i$. A general solution of the associated homogeneous equation is $y_h(x) = e^{-4x}(C_1 \cos 5x + C_2 \sin 5x)$. By operators,

$$\begin{aligned}
y_p &= \frac{1}{2D^2+16D+82}(-2e^{2x}\sin x) = -\frac{1}{D^2+8D+41}\mathrm{Im}[e^{(2+i)x}] = -\mathrm{Im}\left[\frac{1}{D^2+8D+41}e^{(2+i)x}\right] \\
&= -\mathrm{Im}\left[e^{(2+i)x}\frac{1}{(D+2+i)^2+8(D+2+i)+41}(1)\right] = -\mathrm{Im}\left[e^{(2+i)x}\frac{1}{60+12i+\cdots}(1)\right] \\
&= -\frac{1}{12}\mathrm{Im}\left[e^{(2+i)x}\frac{1}{5+i}\frac{5-i}{5-i}\right] = \frac{-1}{12}\mathrm{Im}\left[e^{(2+i)x}\frac{5-i}{26}\right] \\
&= \frac{-1}{312}\mathrm{Im}\big[e^{2x}(\cos x + i \sin x)(5-i)\big] = -\frac{e^{2x}}{312}(-\cos x + 5 \sin x).
\end{aligned}$$

By undetermined coefficients, $y_p = Ae^{2x}\sin x + Be^{2x}\cos x$. Substitution into the differential equation gives

$$\begin{aligned}
&2(4Ae^{2x}\sin x + 4Ae^{2x}\cos x - Ae^{2x}\sin x + 4Be^{2x}\cos x - 4Be^{2x}\sin x - Be^{2x}\cos x) \\
&\quad + 16(2Ae^{2x}\sin x + Ae^{2x}\cos x + 2Be^{2x}\cos x - Be^{2x}\sin x) \\
&\quad + 82(Ae^{2x}\sin x + Be^{2x}\cos x) = -2e^{2x}\sin x.
\end{aligned}$$

When we equate coefficients of $e^{2x}\sin x$ and $e^{2x}\cos x$:

$$120A - 24B = -2, \qquad 120B + 24A = 0.$$

These imply that $A = -5/312$ and $B = 1/312$, and once again $y_p = e^{2x}(\cos x - 5\sin x)/312$. A general solution of the differential equation is therefore

$$y(x) = e^{-4x}(C_1 \cos 5x + C_2 \sin 5x) + e^{2x}(\cos x - 5 \sin x)/312.$$

3.

The auxiliary equation is $0 = 2m^3 - 6m^2 - 12m + 16 = 2(m-1)(m-4)(m+2)$ with solutions $m = 1, -2, 4$. A general solution of the associated homogeneous equation is $y_h(x) = C_1 e^x + C_2 e^{-2x} + C_3 e^{4x}$. Undetermined coefficients suggests $y_p(x) = Ax^2e^x + Bxe^x + Cx^3 + Dx^2 + Ex + F + G\cos x + H \sin x$.